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The relativistic particle as a mechanical system with non-holonomic constraints

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Abstract
The geometric theory of non-holonomic systems on fibred manifolds is applied to describe the motion of a particle within the theory of special relativity. General motion equations for material particles subjected to potential forces are found. They cover, as particular cases, standard motion equations as well as a generalization of the special relativity theory proposed by Dicke. Moreover, they offer new possibilities for studying the dynamics of relativistic particles interacting with an electromagnetic and/or a scalar field.

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1. Introduction

Classical (time-dependent) mechanical systems represent typical examples to be mathematically modelled on fibred manifolds over a one-dimensional basis—the time axis, and copies of the configuration space as the fibres. On the other hand, in relativistic mechanics the time variable has no physically preferred role. This is the reason why it is believed that fibred manifolds are not quite appropriate underlying geometrical structures for such 'non-parametrized' problems, and why some authors suggest to replace fibred manifolds by manifolds of contact elements (cf [1, 2]).

However, bearing in mind recent results of the geometric theory of non-holonomic mechanical systems on fibred manifolds (cf, e.g., [3–14]), and taking into account the relativistic condition

$$g_{\sigma\nu}\, u^{\sigma} u^{\nu} = 1 \tag{1}$$

on the 4-velocity of a material particle, strongly suggests the idea of understanding the relativistic particle as a good example of a non-holonomic mechanical system on an (appropriately chosen) fibred manifold. The aim of this paper is to study this possibility, and to provide explicit calculations. As a result we obtain general motion equations for a material relativistic particle, moving in an electromagnetic field and in the field of a scalar potential. These equations naturally include the so-called *induced constraint force*. We discuss the meaning of this force in different formulations of the special relativity theory. In particular, it appears that the standard motion equations come from the requirement that the induced constraint force vanishes identically. On the other hand, the case of a non-zero induced constraint force is in accordance with a generalization of the relativity theory due to Dicke [15]. Thus, the understanding of the relativistic particle as a non-holonomic system on a fibred manifold provides a unified framework for both the standard and Dicke formulation of the relativity theory. Moreover, it provides us with new possibilities of interaction of relativistic particles with external fields. One of the most interesting consequences is the fact that a scalar field has a direct influence on the electromagnetic interaction.

2. Geometric theory of mechanical systems with non-holonomic constraints

In this section we recall briefly basic notions and results of the general theory of mechanical systems with non-holonomic constraints as formulated in [5, 6].

Let $\pi : Y \to X$ be a fibred manifold with a one-dimensional base X, and let $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$ denote its first and second jet prolongation, respectively. Other commonly used projections are $\pi_{1,0} : J^1Y \to Y$, $\pi_{2,0} : J^2Y \to Y$ and $\pi_{2,1} : J^2Y \to J^1Y$. Let $\dim Y = m + 1$ (i.e. $m \geqslant 1$ the fibre dimension). Denote by (V, ψ), $\psi = (t, q^\sigma)$, where $1 \leqslant \sigma \leqslant m$, a fibred chart on Y, and by (V_1, ψ_1), $\psi_1 = (t, q^\sigma, \dot{q}^\sigma)$, where $V_1 = \pi_{1,0}^{-1}(V)$ (respectively, (V_2, ψ_2), $\psi_2 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$ where $V_2 = \pi_{2,0}^{-1}(V)$) is the associated fibred chart on J^1Y (respectively, J^2Y). By a *section* of the fibred manifold π we shall mean a mapping $\gamma : X \to Y$, defined on an open subset I in X, such that $\pi \circ \gamma$ is the identity mapping of I. The first (respectively, second) jet prolongation of a section γ of π is denoted by $J^1\gamma$ (respectively, $J^2\gamma$); it is a section of the fibred manifold π_1 (respectively, π_2). If γ is in a fibred chart represented by $\gamma = (\gamma^0, \gamma^\sigma)$ where $\gamma^0(t) = t$, $\gamma^\sigma(t) = q^\sigma \circ \gamma(t)$, then $J^1\gamma$ (respectively, $J^2\gamma$) is represented by $J^1\gamma = (\gamma^0, \gamma^\sigma, \bar{\gamma}^\sigma)$ (respectively, $J^2\gamma = (\gamma^0, \gamma^\sigma, \bar{\gamma}^\sigma, \bar{\bar{\gamma}}^\sigma)$), where

$$\bar{\gamma}^\sigma \equiv \dot{q}^\sigma \circ J^1\gamma = \frac{\mathrm{d}\gamma^\sigma}{\mathrm{d}t} = \frac{\mathrm{d}(q^\sigma \circ \gamma)}{\mathrm{d}t} \qquad \bar{\bar{\gamma}}^\sigma \equiv \ddot{q}^\sigma \circ J^2\gamma = \frac{\mathrm{d}\bar{\gamma}^\sigma}{\mathrm{d}t} = \frac{\mathrm{d}^2\gamma^\sigma}{\mathrm{d}t^2} = \frac{\mathrm{d}^2(q^\sigma \circ \gamma)}{\mathrm{d}t^2}.$$

A section δ of π_1 is called *holonomic* if it is the 1-jet prolongation of a section γ of π, i.e. $\delta = J^1\gamma$.

Recall that a vector field ξ on Y is called *projectable* if it projects onto a vector field on the base X. A projectable vector field ξ on Y has the chart expression

$$\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma}$$

i.e. the component at $\partial/\partial t$ is a function of the base coordinate t only. The *first jet prolongation* of a projectable vector field ξ is a vector field on J^1Y which in fibred coordinates reads as follows:

$$J^1\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma} + \left(\frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \frac{\mathrm{d}\xi^0}{\mathrm{d}t}\dot{q}^\sigma\right)\frac{\partial}{\partial \dot{q}^\sigma}.$$

A vector field on Y or on J^1Y is called *vertical* if its projection onto the base is the zero vector field (i.e. $\xi^0 = 0$). A form η on J^1Y is called *contact* if for every section γ of π, $J^1\gamma^*\eta = 0$. Similarly, a form η on J^2Y is called *contact* if for every section γ of π, $J^2\gamma^*\eta = 0$. Every k-form for $k > 1$ is contact. A form is called *horizontal*, if its contraction by an arbitrary vertical vector field vanishes. A 2-form is called 1-*contact* (respectively, 2-*contact*), if its contraction by vertical vector fields is horizontal (respectively, contact). The 1-forms

$$\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \, \mathrm{d}t \qquad \dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma \, \mathrm{d}t$$

$1 \leqslant \sigma \leqslant m$, are contact. Note that besides the standard basis of 1-forms on J^1Y (respectively, J^2Y), i.e. $(\mathrm{d}t, \mathrm{d}q^\sigma, \mathrm{d}\dot{q}^\sigma)$ (respectively, $(\mathrm{d}t, \mathrm{d}q^\sigma, \mathrm{d}\dot{q}^\sigma, \mathrm{d}\ddot{q}^\sigma)$) one has the basis $(\mathrm{d}t, \omega^\sigma, \mathrm{d}\dot{q}^\sigma)$ (respectively, $(\mathrm{d}t, \omega^\sigma, \dot{\omega}^\sigma, \mathrm{d}\ddot{q}^\sigma)$), adapted to the contact structure. Every k-form is then generated by forms of this basis by means of the exterior product. In particular, a 1-form η on J^1Y can be written as

$$\eta = \eta_0 \, \mathrm{d}t + \eta_\sigma \, \mathrm{d}q^\sigma + \tilde{\eta}_\sigma \, \mathrm{d}\dot{q}^\sigma = (\eta_0 + \eta_\sigma \dot{q}^\sigma) \, \mathrm{d}t + \eta_\sigma \omega^\sigma + \tilde{\eta}_\sigma \mathrm{d}\dot{q}^\sigma$$

and it can be uniquely decomposed into the sum of its horizontal and contact component in the following way:

$$\pi_{2,1}^* \eta = (\eta_0 + \eta_\sigma \dot{q}^\sigma + \tilde{\eta}_\sigma \ddot{q}^\sigma) \, \mathrm{d}t + \eta_\sigma \omega^\sigma + \tilde{\eta}_\sigma \, \dot{\omega}^\sigma.$$

A *distribution* on J^1Y is defined as a mapping $\mathcal{D} : J^1Y \ni x \to \mathcal{D}(x) \subset T_x J^1Y$, associating to every point $x \in J^1Y$ a vector subspace $\mathcal{D}(x)$ of the tangent space at x. We say that a distribution $\mathcal{D}$ has a *constant rank* if the dimension of the subspaces $\mathcal{D}(x)$ is a constant (independent of x). A distribution is spanned by a system of local vector fields (ξ_ι), $\iota \in \mathcal{I}$. Alternatively, it can be characterized by a system of 1-forms (η_κ), $\kappa \in \mathcal{K}$, such that $\eta_\kappa(\xi_\iota) = 0$ for every ι and κ. We write

$$\mathcal{D} = \operatorname{span}\{\xi_\iota | \iota \in \mathcal{I}\} \qquad \mathcal{D}^0 = \operatorname{span}\{\eta_\kappa | \kappa \in \mathcal{K}\}$$

and call $\mathcal{D}^0$ the *annihilator* of $\mathcal{D}$. A section δ of π_1 is called an *integral section* of $\mathcal{D}$ if $\delta^*\eta = 0$ for every 1-form η belonging to $\mathcal{D}^0$.

Now, let us turn to the concept of a mechanical system on a fibred manifold, and to a geometrical description of its dynamics by means of distributions.

Let us consider a *dynamical form* on J^2Y, i.e. a 2-form E which in every fibred chart on J^2Y is expressed as follows:

$$E = E_\sigma(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu) \, \mathrm{d}q^\sigma \wedge \mathrm{d}t.$$

A section γ of the fibred manifold π is called a *path* of E if $E \circ J^2\gamma = 0$. Writing down the equations for paths in fibred coordinates we obtain a system of m second-order ordinary differential equations

$$E_\sigma \left(t, \gamma^\nu, \frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t}, \frac{\mathrm{d}^2\gamma^\nu}{\mathrm{d}t^2} \right) = 0 \tag{2}$$

for the components $\gamma^\nu(t)$, $1 \leqslant \nu \leqslant m$, of sections γ of π. Hence, we can see that dynamical forms on fibred manifolds are *global* objects which enable us to transform the (local) concept of motion equations to manifolds. Note that equations (2) are very general second-order ODE: they cover both Lagrangian and non-Lagrangian equations; moreover, they need not admit an expression in the form of a second-order vector field (i.e. they may be 'non-solvable' with respect to the second derivatives). For the purposes of this paper, we can restrict to the case of equations which are affine in accelerations $\ddot{q}^\nu$, i.e. such that

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu \qquad 1 \leqslant \sigma, \nu \leqslant m \tag{3}$$

where A_σ and $B_{\sigma\nu}$ are functions depending on $(t, q^\rho, \dot{q}^\rho)$. Instead of E (which is of the second order) we consider a first-order object—an *equivalence class of local 2-forms on* J^1Y, called the *Lepage class of* E, denoted by $[\alpha]$, and defined as follows: $\alpha \in [\alpha]$ if and only if the one-contact part of α equals E. In fibred coordinates where E is given by (3) we have

$$\alpha = A_\sigma \omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\omega^\sigma \wedge \mathrm{d}\dot{q}^\nu + F$$

where F is a local 2-contact 2-form on J^1Y, i.e. $F = F_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu$. To every α there is naturally associated the so-called *dynamical distribution* Δ_α, generated by all the 1-forms $i_\xi\alpha$, where ξ runs over the set of all π_1-vertical vector fields on J^1Y. In other words, the annihilator of Δ_α is spanned by the following 1-forms:

$$A_\sigma\,\mathrm{d}t + 2F_{\sigma\nu}\omega^\nu + B_{\sigma\nu}\,\mathrm{d}\dot{q}^\nu \qquad B_{\sigma\nu}\omega^\nu.$$

Apparently, the following assertion holds:

Proposition 2.1. *The system of paths of a dynamical form E coincides with the system of holonomic integral sections of its dynamical distributions in the class* $[\Delta_\alpha]$.

The Lepage class of E is called a *first-order mechanical system* (associated with the dynamical form E).

The above rather abstract understanding of a mechanical system first proposed in [5] can be viewed as an appropriate generalization of the *symplectic* description of mechanical systems. While the latter is applicable to regular Lagrangian mechanics on tangent spaces, the former is well adapted to the case when one needs to take into account mechanical systems without *a priori* restrictions (e.g. to regularity or variationality), and to study geometric properties of solutions of the equations of motion on fibred manifolds. This is, in particular, important in the case when constraints have to be considered, since generally neither regularity nor variationality in its usual sense is preserved under non-holonomic constraints.

Naturally, within the general scheme, *regular* and *Lagrangian* systems appear as particular cases: a mechanical system $[\alpha]$ is called *regular*, if the class $[\Delta_\alpha]$ contains a dynamical distribution of rank one. It can be shown [5] that

Proposition 2.2. *The mechanical system* $[\alpha]$ *associated with a dynamical form E on* J^2Y, $E = (A_\sigma + B_{\sigma\nu}\ddot{q}^\nu)\omega^\sigma \wedge \mathrm{d}t$, *is regular if and only if any of the following equivalent conditions holds:*

(a) the matrix $(B_{\sigma\nu})$ *is everywhere regular;*
(b) any dynamical distribution $\Delta \in [\Delta_\alpha]$ *is of rank 1;*
(c) all the dynamical distributions belonging to the class $[\Delta_\alpha]$ *coincide;*
(d) the equations for the paths of E can be expressed in the explicit form

$$\ddot{q}^\sigma = -B^{\sigma\nu}A_\nu \qquad 1 \leqslant \sigma \leqslant m$$

where $(B^{\sigma\nu})$ *denotes the inverse matrix to* $(B_{\sigma\nu})$.

A dynamical form E is called *locally variational* if in a neighbourhood V_x of every point $x \in J^1Y$ there exists a function L such that, over V_x, the components E_σ of E coincide with the Euler–Lagrange expressions of L, i.e.

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\sigma}. \tag{4}$$

It can be proved [5, 16] that

Proposition 2.3. *E is locally variational if and only if, in a neighbourhood* V_x *of every point* $x \in J^1Y$, *the corresponding mechanical system* $[\alpha]$ *contains a closed 2-form* α_E, *i.e. on* V_x, *there is a form* $\alpha_E \in [\alpha]$ *such that* $\mathrm{d}\alpha_E = 0$. *In this case, moreover, the form* α_E *is unique and globally defined (on* J^1Y*).*

The form α_E is called the *Lepagean equivalent* of E and the corresponding mechanical system is called a *Lagrangian system* [16].

Now, let us turn to constrained systems. Let $1 \leqslant k \leqslant m-1$. By a *constraint manifold* in J^1Y we shall mean a submanifold $\mathcal{Q}$ of J^1Y of codimension k, fibred over Y; we denote by ι the canonical embedding of $\mathcal{Q}$ into J^1Y. By definition, $\mathcal{Q}$ is locally defined by a system of k equations, called *non-holonomic constraints*, as follows:

$$f^i(t, q^\sigma, \dot{q}^\sigma) = 0 \qquad \operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k \qquad 1 \leqslant i \leqslant k. \tag{5}$$

Equations (5) can always locally be put into the so-called *normal form*,

$$\dot{q}^{m-k+i} - h^i(t, q^\sigma, \dot{q}^l) = 0 \qquad 1 \leqslant i \leqslant k$$

where $1 \leqslant l \leqslant m-k$. Thus, without loss of generality, one can suppose one has a cover $\mathcal{A}$ of the constraint $\mathcal{Q}$ by open (in J^1Y) sets such that at each of these sets,

$$f^i(t, q^\sigma, \dot{q}^\sigma) = \dot{q}^{m-k+i} - h^i(t, q^\sigma, \dot{q}^l). \tag{6}$$

Consequently, one has for every $U \in \mathcal{A}$ a system of k linearly independent 1-forms,

$$\varphi^i = f^i\,\mathrm{d}t + \frac{\partial f^i}{\partial \dot{q}^\sigma}\,\omega^\sigma = \left(f^i - \frac{\partial f^i}{\partial \dot{q}^\sigma}\dot{q}^\sigma\right)\mathrm{d}t + \frac{\partial f^i}{\partial \dot{q}^\sigma}\,\mathrm{d}q^\sigma \qquad 1 \leqslant i \leqslant k$$

defined on U, and called *constraint 1-forms*. The distribution of corank $2k$ on U annihilated by the constraint 1-forms φ^i and the 1-forms $\mathrm{d}f^i$, $1 \leqslant i \leqslant k$, is then called a *constraint distribution*, and is denoted by $\mathcal{C}_U$.

Denote by $\mathcal{I}(\varphi^i)$ the ideal generated by the constraint 1-forms φ^i, $1 \leqslant i \leqslant k$, on U. A dynamical form Φ_U on U is called a *constraint force* or *Chetaev force* if $\Phi_U \in \mathcal{I}(\varphi^i)$. Thus we have

$$\Phi_U = \lambda_i \wedge \varphi^i = -\lambda_{i0}\,\frac{\partial f^i}{\partial \dot{q}^\sigma}\,\mathrm{d}q^\sigma \wedge \mathrm{d}t.$$

The horizontal 1-forms $\lambda_i = \lambda_{i0}\,\mathrm{d}t$ (respectively, the functions λ_{i0}) are called *Lagrange multipliers*. Every Chetaev force satisfies the *principle of virtual work*:

Proposition 2.4. *For every π_1-vertical vector field belonging to the constraint distribution,*

$$i_\xi \Phi_U = 0.$$

A pair $(\mathcal{Q}, \Phi_U)$ where $\mathcal{Q} \subset J^1Y$ is a constraint and Φ_U is a Chetaev force is called a *physical constraint structure* on U. Considering this structure at each U of the covering $\mathcal{A}$ reflects the physical requirement that the constraint $\mathcal{Q}$ is *ideal* (*workless*). If $[\alpha]$ is a mechanical system on J^1Y we put

$$\alpha_{\Phi_U} = \alpha + \Phi_U.$$

In this way we get at each U a new mechanical system $[\alpha_{\Phi_U}]$, called a *deformation* of $[\alpha]$ by Φ_U. If the corresponding (unconstrained) dynamical form is $E = (A_\sigma + B_{\sigma\nu}\ddot{q}^\nu)\,\mathrm{d}q^\sigma \wedge \mathrm{d}t$, the $f^i = 0$ are equations of the submanifold $\mathcal{Q} \cap U$, and Φ_U is a Chetaev force, then

$$[\alpha_{\Phi_U}] = \left(A_\sigma - \lambda_{i0}\,\frac{\partial f^i}{\partial \dot{q}^\sigma}\right)\omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\omega^\sigma \wedge \mathrm{d}\dot{q}^\nu + F$$

where F runs over 2-contact 2-forms on U. The *deformed equations of motion* on U are the following:

Proposition 2.5. *Let γ be a section of π such that $J^1\gamma$ is an integral section of the constraint distribution $\mathcal{C}_U$. γ is a path of the deformed mechanical system $[\alpha_{\Phi_U}]$ on U iff*

$$J^1\gamma^* i_\xi \alpha_{\Phi_U} = 0$$

for every π_1-vertical vector field ξ on U, where α_{Φ_U} is any representative of the class $[\alpha_{\Phi_U}]$.

In fibred coordinates,

$$f^i \circ J^1\gamma = 0 \qquad A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = \lambda_{i0}\frac{\partial f^i}{\partial \dot{q}^\sigma} \qquad \text{along} \quad J^2\gamma. \tag{7}$$

Thus, locally, we have a system of $m+k$ ODEs for $m+k$ unknowns $\gamma^\sigma(t)$, $\lambda_{i0}(t)$ from which both the constraint force and the constrained dynamics can be determined.

The above 'physical description' gives us constrained systems modelled as *local deformations* of the original mechanical systems, defined in a neighbourhood of the constraint. Another possibility (expressing a 'geometrical point of view') is based on the idea of representing constrained systems directly as mechanical systems *on the constraint submanifold*, i.e. with a reduced number of degrees of freedom (equal to $m-k$). In fact, this is a geometric procedure of extracting Lagrange multipliers, since in the 'reduced' equations of motion no undetermined constraint forces appear.

Denote by $\iota : \mathcal{Q} \to J^1Y$ the canonical embedding and put

$$\mathcal{C}^0 = \mathrm{span}\{\iota^*\varphi^i, 1 \leqslant i \leqslant k\}$$

where φ^i runs over all constraint 1-forms subordinate to a cover of $\mathcal{Q}$. It can be proved that

Proposition 2.6. *$\mathcal{C}$ is a distribution on the constraint $\mathcal{Q}$ (of corank k with respect to $\mathcal{Q}$).*

It is called the *canonical distribution* or *Chetaev bundle* [5]. Clearly, it is a geometric realization of the concept of 'possible general displacements'; its π_1-vertical subdistribution represents 'virtual generalized displacements', and its $\pi_{1,0}$-vertical subdistribution could be called a distribution of 'virtual velocities'. The pair $(\mathcal{Q}, \mathcal{C})$ is then called a *geometrical constraint structure* on J^1Y. The ideal on $\mathcal{Q}$ generated by the 1-forms annihilating the canonical distribution is called the *constraint ideal* and is denoted by $\mathcal{I}(\mathcal{C}^0)$. Now, with the help of the canonical distribution $\mathcal{C}$ one can obtain an intrinsic description of mechanical systems constrained to a submanifold $\mathcal{Q}$ of J^1Y as follows: if $[\alpha]$ is an unconstrained mechanical system, put for every $\alpha \in [\alpha]$

$$\alpha_\mathcal{Q} = \iota^*\alpha \mod \mathcal{I}(\mathcal{C}^0).$$

Thus $\alpha_\mathcal{Q}$ is an equivalence class of 2-forms *on the constraint $\mathcal{Q}$*. We denote by $[\alpha_\mathcal{Q}]$ the mechanical system generated by $\alpha_\mathcal{Q}$. It is easy to see that if $\alpha_1, \alpha_2 \in [\alpha]$ then $[(\alpha_1)_\mathcal{Q}] = [(\alpha_2)_\mathcal{Q}]$. The class $[\alpha_\mathcal{Q}]$ is called the *constrained system* related to the mechanical system $[\alpha]$ and the constraint structure $(\mathcal{Q}, \mathcal{C})$. Note that, by definition, a form belongs to $[\alpha_\mathcal{Q}]$ if and only if it is a sum of $\iota^*\alpha$, a 2-contact 2-form and a constraint 2-form. By the following theorem the above 'physical' and 'geometric' description of a constrained system are equivalent:

Proposition 2.7. *The constrained system $[\alpha_\mathcal{Q}]$ does not depend upon deformation of $[\alpha]$, i.e. on $\mathcal{Q} \cap U$, for every constraint force Φ_U,*

$$[(\alpha_{\Phi_U})_\mathcal{Q}] = [\alpha_\mathcal{Q}].$$

After some technical calculations we obtain a representative α_Q of the class $[\alpha_Q]$ expressed in fibred coordinates in the basis $(\mathrm{d}t, \omega^l, \mathrm{d}\dot{q}^l)$, $1 \leqslant l \leqslant m-k$, as follows (see [5,6]):

$$\alpha_Q = \sum_{l=1}^{m-k} \overline{A}_l \omega^l \wedge \mathrm{d}t + \sum_{l,s=1}^{m-k} \overline{B}_{ls} \omega^l \wedge \mathrm{d}\dot{q}^s$$

where

$$\overline{A}_l = \left(A_l + \sum_{p=1}^{k} A_{m-k+p} \frac{\partial h^p}{\partial \dot{q}^l} + \sum_{s=1}^{k} \left(B_{l,m-k+s} + \sum_{j=1}^{k} B_{m-k+j,m-k+s} \frac{\partial h^j}{\partial \dot{q}^l} \right) \left(\frac{\partial h^s}{\partial t} + \frac{\partial h^s}{\partial q^\sigma} \dot{q}^\sigma \right) \right) \circ \iota \tag{8}$$

$$\overline{B}_{ls} = \left(B_{ls} + \sum_{r=1}^{k} \left(B_{l,m-k+r} \frac{\partial h^r}{\partial \dot{q}^s} + B_{m-k+r,s} \frac{\partial h^r}{\partial \dot{q}^l} \right) + \sum_{r,j=1}^{k} B_{m-k+j,m-k+r} \frac{\partial h^j}{\partial \dot{q}^l} \frac{\partial h^r}{\partial \dot{q}^s} \right) \circ \iota. \tag{9}$$

Now, the *equations of motion* have the following form:

Proposition 2.8. *A section γ of π is a path of the constrained system $[\alpha_Q]$ if and only if $J^1\gamma$ is an integral section of the canonical distribution $\mathcal{C}$, and for every π_1-vertical vector field $\xi \in \mathcal{C}$ it satisfies the equation*

$$J^1\gamma^* i_\xi \alpha_Q = 0 \tag{10}$$

where α_Q is (any) 2-form belonging to the equivalence class $[\alpha_Q]$.

In fibre coordinates this gives the following system of $m-k$ second-order ODEs, and k first-order ODEs for the components $\gamma^1, \dots \gamma^m$ of γ:

$$\left(\overline{A}_l + \sum_{p=1}^{m-k} \overline{B}_{lp} \ddot{q}^p \right) \circ J^2\gamma = 0 \qquad f^i \circ J^1\gamma = 0. \tag{11}$$

Equivalently, motion equations can be interpreted within the differential systems approach as equations for holonomic integral sections of a *constrained dynamical distribution* Δ_{α_Q} associated with the mechanical system represented by α_Q, which is defined to be a subdistribution of the canonical distribution $\mathcal{C}$, generated by 1-forms $i_\xi \alpha_Q$, with ξ running over the set of all π_1-vertical vector fields belonging to $\mathcal{C}$. In analogy with the unconstrained case, this leads to a concept of *regularity* for constrained systems as follows [5]: a constrained system is called *regular* if it is represented by a constrained dynamical distribution of rank 1.

Proposition 2.9. *Regularity is equivalent with the condition that the $(m-k) \times (m-k)$-matrix $(\overline{B}_{sl})$ is regular.*

Note that a constrained system corresponding to a regular mechanical system *need not* be regular. Moreover, a constrained system of a Lagrangian (variational) system need not be variational in the standard sense.

3. The relativistic particle as a non-holonomic mechanical system: the case of constant rest mass

In this section we show that a particle in the theory of special relativity can be viewed as a typical constrained mechanical system with one nonlinear non-holonomic constraint.

Let us suppose that a relativistic particle with the rest mass $m_0 = \text{constant} > 0$ moves in a variational force field. The appropriate underlying fibred manifold for its description is $\boldsymbol{R} \times \boldsymbol{R}^4 \to \boldsymbol{R}$ with the base $\boldsymbol{R}$, playing the role of a space of parameters (parametrizing spacetime curves in $\boldsymbol{R}^4$), and the four-dimensional copies of $\boldsymbol{R}^4$ as the fibres; we consider the manifold $\boldsymbol{R}^4$ with the standard structure of spacetime in the special relativity theory, i.e. endowed by the Minkowski metric $g = (g_{\sigma\nu})$, where $g_{lp} = -\delta_{lp}$ and $g_{l4} = g_{4l} = 0$ for $1 \leqslant l, p \leqslant 3$, and $g_{44} = 1$. In (global) canonical fibred coordinates $(s, q^\sigma, \dot{q}^\sigma)$ on $J^1(\boldsymbol{R} \times \boldsymbol{R}^4) = \boldsymbol{R} \times \boldsymbol{R}^4 \times \boldsymbol{R}^4$ the corresponding Lagrange function has the following form:

$$L = -\frac{1}{2} m_0 \left[(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 \right] + \dot{q}^\sigma \phi_\sigma - \psi \tag{12}$$

where $1 \leqslant p \leqslant 3$, $1 \leqslant \sigma \leqslant 4$, ϕ_σ and ψ are C^1-differentiable functions depending on (q^l, q^4), $1 \leqslant l \leqslant 3$, but not on the base parameter s. The corresponding equations of motion are the Euler–Lagrange equations of the Lagrangian (12); they are second-order ODE for sections γ of the fibred manifold $\boldsymbol{R} \times \boldsymbol{R}^4 \to \boldsymbol{R}$,

$$E_l \circ J^2\gamma = 0 \qquad E_4 \circ J^2\gamma = 0 \qquad 1 \leqslant l \leqslant 3$$

where the Euler–Lagrange expressions $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu$ are of the form

$$E_l = -m_0 \ddot{q}^l + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_l}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^l} \qquad E_4 = m_0 \ddot{q}^4 + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^4} - \frac{\partial \phi_4}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^4}.$$

Thus, by (3), we have

$$A_l = \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_l}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^l} \quad 1 \leqslant l \leqslant 3 \qquad A_4 = \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^4} - \frac{\partial \phi_4}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^4} \tag{13}$$

$$B_{lp} = -m_0 \delta_{lp} \qquad B_{4l} = B_{l4} = 0 \qquad B_{44} = m_0 \qquad 1 \leqslant l, p \leqslant 3. \tag{14}$$

In keeping with the special relativity theory, we shall consider one single non-holonomic constraint ($k = 1$) in $J^1(\boldsymbol{R} \times \boldsymbol{R}^4)$, given by the equation

$$(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 - 1 = 0. \tag{15}$$

This equation defines a smooth manifold $\mathcal{Q}$ in $J^1(\boldsymbol{R} \times \boldsymbol{R}^4)$ of codimension one. Apparently, $\mathcal{Q}$ is not connected, being the union of two connected components $\mathcal{Q}_+ = \{(s, q^\sigma, \dot{q}^\sigma) \in \mathcal{Q} \mid \dot{q}^4 > 0\}$ and $\mathcal{Q}_- = \{(s, q^\sigma, \dot{q}^\sigma) \in \mathcal{Q} \mid \dot{q}^4 < 0\}$. We shall consider the physically relevant component $\mathcal{Q}_+$ given by the equation

$$\dot{q}^4 = \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{16}$$

Note that the open set $U = \{x \in J^1(\boldsymbol{R} \times \boldsymbol{R}^4) \mid \dot{q}^4(x) > 0\}$ with the canonical fibred coordinates is a *global* chart covering the constraint manifold $\mathcal{Q}_+$. In the notation of section 2 put

$$f = f^1 = \dot{q}^4 - \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \qquad h = h^1 = \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{17}$$

On U we obtain the constraint ideal generated by the 1-form

$$\varphi = \left(\dot{q}^4 - \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}\right) \mathrm{d}s - \sum_{l=1}^{3} \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}\, \omega^l + \omega^4$$

and the corresponding Chetaev force depending upon one Lagrange multiplier $\lambda_0\, \mathrm{d}s$,

$$\Phi_U = \lambda_0 \sum_{l=1}^{3} \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}\, \omega^l \wedge \mathrm{d}s - \lambda_0\, \omega^4 \wedge \mathrm{d}s.$$

The *motion equations of the deformed mechanical system* now become

$$-m_0 \ddot{q}^l + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_l}{\partial q^\sigma}\right) - \frac{\partial \psi}{\partial q^l} = \frac{-\lambda_0 \dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} \tag{18}$$

$$m_0 \ddot{q}^4 + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^4} - \frac{\partial \phi_4}{\partial q^\sigma}\right) - \frac{\partial \psi}{\partial q^4} = \lambda_0. \tag{19}$$

Let us express the constrained system as a *mechanical system on the constraint manifold* $\mathcal{Q}_+$. The canonical embedding of $\mathcal{Q}_+$ into the first jet prolongation of the underlying fibred manifold has the form

$$\iota : \mathcal{Q}_+ \ni (s, q^l, q^4, \dot{q}^l) \to \iota(s, q^l, q^4, \dot{q}^l) = \left(s, q^l, q^4, \dot{q}^l, \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}\right) \in J^1 Y.$$

Using the relations (8) and (9) we obtain the functions $(\overline{A}_l, \overline{B}_{ls})$, $1 \leqslant l, s \leqslant 3$, the components of the 2-form $\alpha_{\mathcal{Q}_+}$ on $\mathcal{Q}_+$:

$$\begin{aligned}\overline{A}_l = \left(A_l + A_4 \frac{\partial h}{\partial \dot{q}^l}\right) \circ \iota = \sum_{j=1}^{3} \dot{q}^j \left(\frac{\partial \phi_j}{\partial q^l} - \frac{\partial \phi_l}{\partial q^j}\right) - \frac{\partial \psi}{\partial q^l} \\ + \left(\dot{q}^j \left(\frac{\partial \phi_j}{\partial q^4} - \frac{\partial \phi_4}{\partial q^j}\right) - \frac{\partial \psi}{\partial q^4}\right) \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} \\ + \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \left(\frac{\partial \phi_4}{\partial q^l} - \frac{\partial \phi_l}{\partial q^4}\right)\end{aligned} \tag{20}$$

$$\overline{B}_{ls} = \left(B_{ls} + \left(B_{l4} \frac{\partial h}{\partial \dot{q}^s} + B_{4s} \frac{\partial h}{\partial \dot{q}^l}\right) + B_{44} \frac{\partial h}{\partial \dot{q}^l} \frac{\partial h}{\partial \dot{q}^s}\right) \circ \iota = -m_0 \left(\delta_{ls} - \frac{\dot{q}^l \dot{q}^s}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}\right). \tag{21}$$

Recall that the 2-form $\alpha_{\mathcal{Q}_+}$ represents the constrained mechanical system, and the components of $\alpha_{\mathcal{Q}_+}$ define the left-hand sides of the corresponding motion equations. Hence, the reduced system of equations of motion is of the form

$$(\overline{A}_l + \overline{B}_{lp} \ddot{q}^p) \circ J^2 \gamma = 0.$$

Since the matrix $(\overline{B}_{lp})$ is regular, we can write the equations of motion in the explicit form

$$\ddot{q}^p + \mathcal{A}^p = 0 \tag{22}$$

with

$$\mathcal{A}^p = \overline{B}^{pl}\overline{A}_l \qquad \overline{B}^{-1} = (\overline{B}^{pl}).$$

The constrained system is described by equations (22) together with the equation of the constraint (16). For the inverse matrix $\overline{B}^{-1} = (\overline{B}^{pl})$ we obtain

$$\overline{B}^{-1} = -\frac{1}{m_0}\begin{pmatrix} 1+(\dot{q}^1)^2 & \dot{q}^1\dot{q}^2 & \dot{q}^1\dot{q}^3 \\ \dot{q}^1\dot{q}^2 & 1+(\dot{q}^2)^2 & \dot{q}^2\dot{q}^3 \\ \dot{q}^1\dot{q}^3 & \dot{q}^2\dot{q}^3 & 1+(\dot{q}^3)^2 \end{pmatrix}$$

i.e.

$$\overline{B}^{pl} = -\frac{1}{m_0}(\delta^{pl} + \dot{q}^p\dot{q}^l) \qquad 1 \leqslant p, l \leqslant 3. \tag{23}$$

After some calculations we obtain the functions $\mathcal{A}^p$ in equations (22):

$$\mathcal{A}^p = \overline{B}^{pl}\,\overline{A}_l = -\frac{1}{m_0}\sum_{l=1}^{3}\left(\delta^{pl}\dot{q}^j\left(\frac{\partial\phi_j}{\partial q^l} - \frac{\partial\phi_l}{\partial q^j}\right) + \delta^{pl}\sqrt{1+\sum_{s=1}^{3}(\dot{q}^s)^2}\left(\frac{\partial\phi_4}{\partial q^l} - \frac{\partial\phi_l}{\partial q^4}\right)\right.$$
$$\left. - \left(\delta^{pl}\frac{\partial\psi}{\partial q^l} + \dot{q}^p\dot{q}^l\frac{\partial\psi}{\partial q^l} + \dot{q}^p\frac{\partial\psi}{\partial q^4}\sqrt{1+\sum_{s=1}^{3}(\dot{q}^s)^2}\right)\right). \tag{24}$$

Let us denote $q^4 = t$ as usual, and consider on $J^1(\boldsymbol{R}\times\boldsymbol{R}^4)$ new coordinates $(s, q^l, t, v^l, \dot{q}^4)$, defined by the transformation rule

$$\dot{q}^l = v^l\dot{q}^4.$$

Note that the meaning of the new coordinates is the following: (t, q^l, v^l) are coordinates on $J^1(\boldsymbol{R}\times\boldsymbol{R}^3)$, adapted to the fibration $\boldsymbol{R}\times\boldsymbol{R}^3 \to \boldsymbol{R}$ of the manifold $\boldsymbol{R}^4$—the fibre of the fibred manifold $\boldsymbol{R}\times\boldsymbol{R}^4 \to \boldsymbol{R}$. In these coordinates the constraint $\mathcal{Q}_+$ is given by the equation

$$\dot{q}^4 = \frac{1}{\sqrt{1-v^2}} \tag{25}$$

where $v^2 = \sum_{j=1}^{3}(v^j)^2 = \sum_{j=1}^{3}(\mathrm{d}q^j/\mathrm{d}t)^2$ is the (three-dimensional) velocity of the particle. Now, equations (22) with the functions $\mathcal{A}^p$ given by (24) can be transformed by eliminating the parameter s as follows: since we have for $p = 1, 2, 3$,

$$\ddot{q}^p = \frac{\mathrm{d}}{\mathrm{d}s}(\dot{q}^p) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}q^p}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}s}\right) = \frac{1}{\sqrt{1-v^2}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^p}{\sqrt{1-v^2}}\right) \tag{26}$$

we obtain

$$\frac{1}{\sqrt{1-v^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0v^p}{\sqrt{1-v^2}}\right) + m_0\mathcal{A}^p = 0 \tag{27}$$

and using (25) we obtain for $(\mathcal{A}^p)$, $1 \leqslant p \leqslant 3$,

$$\mathcal{A}^p = -\frac{\delta^{pl}}{m_0\sqrt{1-v^2}}\left(v^j\left(\frac{\partial\phi_j}{\partial q^l} - \frac{\partial\phi_l}{\partial q^j}\right) + \left(\frac{\partial\phi_4}{\partial q^l} - \frac{\partial\phi_l}{\partial t}\right)\right)$$
$$+\frac{1}{m_0(1-v^2)}\left((\delta^{pl}(1-v^2) + v^pv^l)\frac{\partial\psi}{\partial q^l} + v^p\frac{\partial\psi}{\partial t}\right).$$

We denote $\vec{A} = (\phi_l)$ and $\phi_4 = -V$. The vector form of equations (27) multiplied by $\sqrt{1-v^2}$ is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2}}\right) = \vec{v} \times \operatorname{rot}\vec{A} - \frac{\partial \vec{A}}{\partial t} - \operatorname{grad} V + \vec{\mathcal{B}} \tag{28}$$

where we have denoted $\vec{v} = (v^l)$, and

$$\begin{aligned}\vec{\mathcal{B}} &= -\sqrt{1-v^2}\operatorname{grad}\psi - \frac{\vec{v}}{\sqrt{1-v^2}}\left(\frac{\partial\psi}{\partial t} + \vec{v}\operatorname{grad}\psi\right)\\ &= -\sqrt{1-v^2}\operatorname{grad}\psi - \frac{\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\psi}{\mathrm{d}t}.\end{aligned} \tag{29}$$

In this way we have obtained the following result. The original problem—to study the mechanical system on $\boldsymbol{R} \times \boldsymbol{R}^4 \to \boldsymbol{R}$, defined by the Lagrangian (12), and subject to the standard relativistic constraint $g_{\sigma\nu}\dot{q}^\sigma\dot{q}^\nu = 1$, has been transferred to an equivalent problem—to study the *unconstrained 'three-dimensional particle'*, moving in the force field

$$\vec{\mathcal{F}} = \vec{\mathcal{F}}_L + \vec{\mathcal{B}}$$

where $\vec{\mathcal{F}}_L$ is the standard Lorentz force, and $\vec{\mathcal{B}}$ is defined by (29). Note that both $\vec{\mathcal{F}}_L$ and $\vec{\mathcal{B}}$ are force fields on $\mathcal{Q}_+$. Consequently the following proposition holds.

Proposition 3.1. *A section γ of the fibred manifold $\boldsymbol{R} \times \boldsymbol{R}^4 \to \boldsymbol{R}$, $\gamma(s) = (s, t(s), q^l(t(s)))$, is a path of a relativistic particle with a constant rest mass $m_0 > 0$ moving in a potential force field given by a 4-potential $(\vec{A}, V)$ and a scalar potential ψ if and only if along $J^2\gamma$ the following equations are satisfied:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2}}\right) = \vec{v} \times \operatorname{rot}\vec{A} - \frac{\partial \vec{A}}{\partial t} - \operatorname{grad} V - \sqrt{1-v^2}\operatorname{grad}\psi - \frac{\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{30}$$

$$\mathcal{E} \equiv m_0\dot{q}^4 = \frac{m_0}{\sqrt{1-v^2}}. \tag{31}$$

The first of these equations is the *equation of motion*, while the second one is the *energy equation*.

Let us discuss the geometric and physical meaning of the force $\vec{\mathcal{B}}$ in more detail.

For $\vec{\mathcal{B}} = \vec{0}$ we obtain the well known vector equation for a particle moving in the variational force field described by a vector potential $\vec{A}$ and scalar potential V. This equation can be derived from the Lagrange function

$$\mathcal{L} = -m_0\sqrt{1-v^2} + \vec{v}\vec{A} - V. \tag{32}$$

It is interesting to note the following:

Proposition 3.2. *The condition $\vec{\mathcal{B}} = \vec{0}$ is a necessary and sufficient condition for the motion equation (30) be variational, i.e. identical with the Euler–Lagrange equation of some Lagrange function.*

Actually, as proved in [17, 18] (cf also [19]), a force $\vec{\mathcal{F}}$ in the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2}}\right) = \vec{\mathcal{F}} \tag{33}$$

is variational (potential) if and only if $\vec{\mathcal{F}}$ is a 'Lorentz-type' force, i.e. iff there exists a vector field $\vec{w}$ and a function φ on the spacetime such that

$$\vec{\mathcal{F}} = \vec{v} \times \operatorname{rot} \vec{w} - \frac{\partial \vec{w}}{\partial t} - \operatorname{grad} \varphi.$$

Comparing this result with (30) we can see that this means that $\vec{w}$ and φ are identified with $\vec{A}$ and V, respectively, and $\vec{\mathcal{B}} = 0$, proving the above assertion.

Let us turn to discuss the case $\vec{\mathcal{B}} \neq \vec{0}$. To investigate the meaning of the second term of $\vec{\mathcal{B}}$, i.e. the force field

$$\vec{\mathcal{F}}_c = -\frac{\vec{v}}{\sqrt{1-v^2}} \frac{\mathrm{d}\psi}{\mathrm{d}t}$$

let us proceed as follows. Consider the deformed equations of motion (18) and (19). We can see that along the paths of the deformed mechanical system,

$$m_0 g_{\sigma\nu} \dot{q}^\sigma \ddot{q}^\nu + \left(\frac{\partial \phi_\nu}{\partial q^\sigma} - \frac{\partial \phi_\sigma}{\partial q^\nu} \right) \dot{q}^\nu \dot{q}^\sigma - \frac{\partial \psi}{\partial q^\sigma} \dot{q}^\sigma = \lambda_0 \frac{g_{\sigma\nu} \dot{q}^\sigma \dot{q}^\nu}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}.$$

The second term on the left-hand side of this equation is zero because of the antisymmetry of the expression in brackets. Since only solutions satisfying the equation of the constraint (15) are admissible, i.e. such that $g_{\sigma\nu} \dot{q}^\sigma \dot{q}^\nu \circ J^1\gamma = 1$ and consequently, $g_{\sigma\nu} \dot{q}^\sigma \ddot{q}^\nu \circ J^2\gamma = 0$, we finally obtain

$$\frac{\partial \psi}{\partial q^\sigma} \dot{q}^\sigma = \frac{\mathrm{d}\psi}{\mathrm{d}s} = -\frac{\lambda_0}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}$$

along admissible paths. Comparing this with the expression for the Chetaev force we can see that

$$\Phi^l = -\dot{q}^l \frac{\mathrm{d}\psi}{\mathrm{d}s}$$

are its (space) components. In the coordinates $(s, t, q^l, v^l, \dot{q}^4)$ this reads

$$\Phi^l = -v^l \frac{\mathrm{d}\psi}{\mathrm{d}t} (\dot{q}^4)^2 = -\frac{v^l}{1-v^2} \frac{\mathrm{d}\psi}{\mathrm{d}t}.$$

Thus we have obtained that along *any admissible* trajectory the term

$$-\frac{v^l}{1-v^2} \frac{\mathrm{d}\psi}{\mathrm{d}t}$$

has the geometric meaning of a constraint force. Therefore, and taking into account that (28) is equation (27) multiplied by the factor $\sqrt{1-v^2}$, we can call $\vec{\mathcal{F}}_c$ the *induced constraint force*. Summarizing, we obtained that *the second term in the expression (29) of* $\vec{\mathcal{B}}$ *is the induced constraint force.*

Now, we shall discuss the meaning of the first term $\vec{\mathcal{D}} = -\sqrt{1-v^2} \operatorname{grad} \psi$ in $\vec{\mathcal{B}}$. To this end we rewrite equation (30) in a slightly different form,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m_0 \vec{v}}{\sqrt{1-v^2}} \right) + \frac{\vec{v}}{\sqrt{1-v^2}} \frac{\mathrm{d}\psi}{\mathrm{d}t} = \vec{v} \times \operatorname{rot} \vec{A} - \frac{\partial \vec{A}}{\partial t} - \operatorname{grad} V - \sqrt{1-v^2} \operatorname{grad} \psi$$

and we suppose for simplicity the electromagnetic force be equal to zero. Put

$$\mu = \frac{1}{m_0} \psi \qquad \widetilde{m}_0 = m_0 e^\mu. \tag{34}$$

In terms of the new scalar potential μ and the function $\widetilde{m}_0$ (which both are functions of the spacetime variables) the above equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}_0 e^{-\mu}\vec{v}}{\sqrt{1-v^2}}\right) + \frac{\widetilde{m}_0 e^{-\mu}\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\mu}{\mathrm{d}t} = -\widetilde{m}_0\, e^{-\mu}\sqrt{1-v^2}\,\mathrm{grad}\,\mu$$

i.e.

$$e^{-\mu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}_0\vec{v}}{\sqrt{1-v^2}}\right) = -\widetilde{m}_0\, e^{-\mu}\sqrt{1-v^2}\,\mathrm{grad}\,\mu.$$

Since $e^{-\mu} \neq 0$, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}_0\vec{v}}{\sqrt{1-v^2}}\right) = -\widetilde{m}_0\sqrt{1-v^2}\,\mathrm{grad}\,\mu = -\sqrt{1-v^2}\,\mathrm{grad}\,\widetilde{m}_0.$$

Accordingly, we can state the following result which is in agreement with a proposal due to Dicke [15] adapting the theory of relativity to the *Mach principle*:

Proposition 3.3. *The motion of a relativistic particle with a constant rest mass $m_0 > 0$, moving in a force field given by a scalar potential ψ is described by the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2}}\right) = -\sqrt{1-v^2}\,\mathrm{grad}\,\psi - \frac{\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{35}$$

or equivalently, by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}_0\vec{v}}{\sqrt{1-v^2}}\right) = -\sqrt{1-v^2}\,\mathrm{grad}\,\widetilde{m}_0 \tag{36}$$

where

$$\widetilde{m}_0 = m_0 e^{\mu} \qquad \mu = \frac{1}{m_0}\psi.$$

Thus this particle can be viewed as possessing a non-constant rest mass $\widetilde{m}_0$ depending upon a scalar potential μ, and subject to the force

$$\vec{\widetilde{\mathcal{D}}} = -\sqrt{1-v^2}\,\mathrm{grad}\,\widetilde{m}_0.$$

We shall call the scalar potential μ the *Dicke field* and the force $\vec{\widetilde{\mathcal{D}}}$ the *Dicke force*. We can see that the Dicke field has the properties assumed by Dicke in [15]: namely, the Dicke force is attractive and proportional to $\sqrt{1-v^2}$. At the same time, the form of the dependence of $\widetilde{m}_0$ upon μ ensures that $\vec{\widetilde{\mathcal{D}}}$ depends upon m_0. Moreover, since $m_0 > 0$, the obtained mass of a particle is everywhere positive. It remains unchanged if and only if the field μ vanishes, and the dependence of the mass upon μ is the same for all particles. These facts agree with the famous Eötvös experiment stating the equivalence of gravitational and inertial mass.

Finally, note that the requirement that the induced constraint force $\vec{\mathcal{F}}_c$ vanishes identically means that $\mathrm{d}\psi/\mathrm{d}t = 0$, i.e. $\psi =$ constant on the constraint. More formally we can write

Proposition 3.4.

$$\vec{\mathcal{F}}_c = 0 \quad \Longleftrightarrow \quad \mathrm{d}\psi/\mathrm{d}t = 0.$$

Indeed, by assumption, ψ is C^1-differentiable, hence $\mathrm{d}\psi/\mathrm{d}t$ is continuous, and, consequently, $\mathrm{d}\psi/\mathrm{d}t(x) \neq 0$ at a point x means that there exists a neighbourhood U of x in $\mathcal{Q}_+$ such that $\mathrm{d}\psi/\mathrm{d}t \neq 0$ on U. Denote by $\mathcal{V}$ the submanifold of $\mathcal{Q}_+$ defined by the equation $\vec{v} = 0$. Obviously, $\mathcal{V}$ is a closed submanifold of codimension three. Since $\vec{\mathcal{F}}_c = 0$ on $\mathcal{Q}_+$, we have $\mathrm{d}\psi/\mathrm{d}t = 0$ on $\mathcal{Q}_+ - \mathcal{V}$. Next, if $x \in \mathcal{V}$ would be such that $\mathrm{d}\psi/\mathrm{d}t(x) \neq 0$ then it should hold $\mathrm{d}\psi/\mathrm{d}t \neq 0$ on an open neighbourhood $U \subset \mathcal{Q}_+$ of x, and due to $\vec{\mathcal{F}}_c = 0$ one would get $\vec{v} = 0$ on U, i.e. $U \subset \mathcal{V}$, a contradiction. Thus $\mathrm{d}\psi/\mathrm{d}t = 0$ on the constraint $\mathcal{Q}_+$. The converse implication, i.e., ψ = constant $\Longrightarrow \vec{\mathcal{F}}_c = 0$ is trivial.

As a direct consequence of the above proposition we obtain that *the condition* $\vec{\mathcal{F}}_c = 0$ *implies that* $\vec{\mathcal{B}} = 0$ *and the motion equations (30) reduce to the 'standard' relativistic motion equations*. Conversely, μ = constant means that the mass of the particle is a constant and the Dicke force vanishes. Hence, we can conclude that approaching the relativistic mechanics as a theory with non-holonomic constraints one gets a unified model for the 'standard' and Dicke relativity theory as follows.

The 'standard' approach (based upon the Lagrange function (32)) corresponds to the assumption that the induced constraint force vanishes. On the other hand, Dicke relativity theory (taking into account the Mach principle) comes from the assumption that the induced constraint force is non-trivial.

Remarkably, the motion of a charged particle in an electromagnetic field is different in presence of a Dicke field.

Proposition 3.5. *In the presence of an electromagnetic field the motion equation (36) takes the form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}_0\vec{v}}{\sqrt{1-v^2}}\right) = e^{\mu}\,\vec{\mathcal{F}}_L + \tilde{\vec{\mathcal{D}}} \tag{37}$$

where $\vec{\mathcal{F}}_L$ *is the standard Lorentz force.*

Thus, the Dicke field can strengthen or weaken the Lorentz force. It seems that interaction of this kind could be responsible, for example, for the stability of atoms.

In what follows we shall call the factor e^{μ} the *cushon charge*, or simply *cushon*.

4. The relativistic particle as a non-holonomic mechanical system: the case of non-constant mass

Now, let us suppose that the mass of a particle is a (general) function of spacetime coordinates. Replacing m_0 in the Lagrangian (12) by $m(q^\lambda) > 0$ we obtain the corresponding Euler–Lagrange expressions $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu$, with

$$A_l = \dot{q}^\sigma\left(\frac{\partial\phi_\sigma}{\partial q^l} - \frac{\partial\phi_l}{\partial q^\sigma}\right) - \frac{\partial\psi}{\partial q^l} - \frac{1}{2}\frac{\partial m}{\partial q^l}\left((\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2\right) - \frac{\partial m}{\partial q^j}\dot{q}^j\dot{q}^l - \frac{\partial m}{\partial q^4}\dot{q}^4\dot{q}^l$$

$$1 \leqslant l \leqslant 3$$

$$A_4 = \dot{q}^\sigma\left(\frac{\partial\phi_\sigma}{\partial q^4} - \frac{\partial\phi_4}{\partial q^\sigma}\right) - \frac{\partial\psi}{\partial q^4} - \frac{1}{2}\frac{\partial m}{\partial q^4}\left((\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2\right) + \frac{\partial m}{\partial q^j}\dot{q}^j\dot{q}^4 + \frac{\partial m}{\partial q^4}(\dot{q}^4)^2$$

$$B_{lp} = -m\delta_{lp} \qquad B_{4l} = B_{l4} = 0 \qquad B_{44} = m \qquad 1 \leqslant l, p \leqslant 3.$$

Since the constraint is again expressed by equation (15) (respectively, (16)), the constraint ideal and the constraint force remain the same as in the case of constant rest mass. Applying the same procedure as in section 3 we obtain the constrained system represented by the motion equations (11) where the functions $\overline{A}_l$ and $\overline{B}_{ls}$, $1 \leqslant l, s \leqslant 3$, are of the following form:

$$\overline{A}_l = \sum_{j=1}^{3} \dot{q}^j \left(\frac{\partial \phi_j}{\partial q^l} - \frac{\partial \phi_l}{\partial q^j} \right) - \frac{\partial \psi}{\partial q^l} + \left(\dot{q}^j \left(\frac{\partial \phi_j}{\partial q^4} - \frac{\partial \phi_4}{\partial q^j} \right) - \frac{\partial \psi}{\partial q^4} \right) \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}$$
$$+ \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \left(\frac{\partial \phi_4}{\partial q^l} - \frac{\partial \phi_l}{\partial q^4} \right) - \frac{1}{2} \left(\frac{\partial m}{\partial q^l} + \frac{\partial m}{\partial q^4} \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} \right)$$

and

$$\overline{B}_{ls} = -m \left(\delta_{ls} - \frac{\dot{q}^l \dot{q}^s}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \right).$$

Analogous calculations such as those presented in section 3 lead to the following vector equation of motion:

$$m \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\vec{v}}{\sqrt{1 - v^2}} \right) = \vec{\mathcal{F}}_L + \vec{\mathcal{B}} + \vec{\mathcal{B}}_m \tag{38}$$

with the standard Lorenz force $\vec{\mathcal{F}}_L$, the $\vec{\mathcal{B}}$ as in section 3 (cf (29)), and with

$$\vec{\mathcal{B}}_m = -\frac{1}{2} \left(\sqrt{1 - v^2} \, \mathrm{grad} \, m + \frac{\vec{v}}{\sqrt{1 - v^2}} \frac{\mathrm{d}m}{\mathrm{d}t} \right). \tag{39}$$

To discuss this equation in more detail, let us rewrite it in the following (equivalent) form

$$m \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\vec{v}}{\sqrt{1 - v^2}} \right) + \frac{\vec{v}}{\sqrt{1 - v^2}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{1}{2} m + \psi \right) = \vec{\mathcal{F}}_L - \sqrt{1 - v^2} \, \mathrm{grad} \left(\tfrac{1}{2} m + \psi \right). \tag{40}$$

Taking into account the deformed equations of motion (7), and applying similar arguments to those in section 3 we obtain the following result:

Proposition 4.1. *The induced constraint force is given by the formula*

$$\vec{\mathcal{F}}_c = -\frac{\vec{v}}{\sqrt{1 - v^2}} \frac{\mathrm{d}\chi}{\mathrm{d}t} \qquad \text{where} \quad \chi = \psi - \tfrac{1}{2} m. \tag{41}$$

Thus, in terms of the constraint force, equation (40) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m \vec{v}}{\sqrt{1 - v^2}} \right) = \vec{\mathcal{F}}_L + \vec{\mathcal{F}}_c - \sqrt{1 - v^2} \, \mathrm{grad}(m + \chi).$$

Now, we can consider two different possibilities:

4.1. Zero induced constraint force.

This case is obtained from the assumption that (up to an additive constant)

$$\psi = \tfrac{1}{2} m.$$

Obviously, equation (40) takes then the 'standard form' of motion equation for a relativistic particle with non-constant mass

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m \vec{v}}{\sqrt{1 - v^2}} \right) = \vec{\mathcal{F}}_L - \sqrt{1 - v^2} \, \mathrm{grad} \, m. \tag{42}$$

Recall that this equation comes as the Euler–Lagrange equation from the Lagrangian

$$\mathcal{L} = -m\sqrt{1-v^2} + \vec{v}\vec{A} - V. \tag{43}$$

Note that equation (36) obtained in the previous section can be viewed as a particular case of (42) if in the latter one takes $m = m_0 e^{\mu}$ and $\vec{\mathcal{F}}_L = 0$. However, for a charged particle moving in an electromagnetic field the corresponding equations (37) and (42) become quite different: in (37) the Lorentz force is modified by the cushon charge e^{μ}.

4.2. Non-zero induced constraint force.

We shall show that the assumption $\vec{\mathcal{F}}_c \neq 0$ leads to a generalized motion equation for a particle with non-constant mass, including equation (37) (and hence also the Dicke relativity) as a particular case.

Consider the motion equation (40). Instead of the functions m and ψ we can equivalently introduce new functions f and μ as follows:

$$m = m_0 + f \qquad \chi = m_0\mu$$

where $m_0 > 0$ is a constant. Rewriting (40) in terms of f and μ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(m_0+f)\vec{v}}{\sqrt{1-v^2}}\right) + \frac{m_0\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \vec{\mathcal{F}}_L - \sqrt{1-v^2}\,\mathrm{grad}(m_0 + f + m_0\mu).$$

Multiplying this equation by the cushon e^{μ} and denoting by

$$\widetilde{m} = me^{\mu} \qquad \widetilde{m}_0 = m_0 e^{\mu} \qquad \widetilde{f} = fe^{\mu} \qquad \widetilde{\vec{\mathcal{F}}}_L = e^{\mu}\vec{\mathcal{F}}_L$$

the corresponding cushon quantities, we obtain after straightforward calculations the following motion equation, equivalent with (40):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widetilde{m}\vec{v}}{\sqrt{1-v^2}}\right) = \widetilde{\vec{\mathcal{F}}}_L - \sqrt{1-v^2}\,\mathrm{grad}\,\widetilde{m} + \widetilde{f}\left(\sqrt{1-v^2}\,\mathrm{grad}\,\mu + \frac{\vec{v}}{\sqrt{1-v^2}}\frac{\mathrm{d}\mu}{\mathrm{d}t}\right). \tag{44}$$

We can see that, as expected, equation (37) is a particular case of (44) for $f = 0$. Similarly, equation (42) is contained in (44), since, by the same arguments as in section 3, $\vec{\mathcal{F}}_c = 0$ implies $\mu = \text{constant}$.

Proposition 4.2. *Within the approach of the theory of non-holonomic systems, equation (44) is the most general motion equation, representing the dynamics of a charged relativistic particle with positive non-constant mass, moving in an electromagnetic field and a Dicke field* μ.

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